Resit exam (online) — Analysis (WPMA14004)

Tuesday 14 April 2020, 19.00h–22.00h CEST (plus 30 minutes for uploading)

University of Groningen

Instructions

1. All answers need to be accompanied with an explanation or a calculation: only answering "yes", "no", or "42" is not sufficient.
2. If p is the number of marks then the exam grade is $G = 1 + p/10$.
3. Write both your name and student number on the answer sheets!
4. This exam comes in two versions. Both versions consist of six problems of equal difficulty.

   Make version 1 if your student number is odd.

   Make version 2 if your student number is even.

   For example, if your student number is 1277456, which is even, then you have to make version 2.
5. Upload your work as a single PDF file in your personal Nestor dropbox folder.

Version 1 (for odd student numbers)

Problem 1 (5 + 5 + 5 = 15 points)

Given an example of each of the following, or argue that such a request is impossible:

(a) A nonempty set with only finitely many elements that does *not* contain its supremum.

(b) A nonempty bounded subset of $\mathbb{Q}$ that contains its infimum but not its supremum.

(c) A nonempty set S with the property $\inf S \geq \sup S$.

Problem 2 (6 + 6 + 3 = 15 points)

Assume that $-1 < r < 1$ and $c \in \mathbb{R}$. Consider the sequence (x_n) defined inductively as follows:

$$x_{n+1} = rx_n + c, \quad n \in \mathbb{N},$$

where $x_1 \in \mathbb{R}$ is arbitrary.

(a) Prove by induction that $x_{n+1} = r^n x_1 + \sum_{k=0}^{n-1} cr^k$ for all $n \in \mathbb{N}$.

(b) Prove that the sequence (x_n) converges.

(c) Compute $\lim x_n$ when $r = -1/3$ and $c = 4$.

Problem 3 (15 points)

Let $A \subseteq \mathbb{R}$ be a nonempty set. Prove that the following set is compact:

$$K = \bigcap_{a \in A} [a-1, a+1].$$

Problem 4 (5 + 5 + 5 = 15 points)

Consider the following functions:

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases} \quad \text{and} \quad g(x) = \begin{cases} x^2 \cos(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

(a) Prove that f and g are differentiable at $x = 0$ and that $f'(0) = g'(0) = 0$.
Hint: only give the proof for f, since the proof for g is similar.

(b) Show that

$$f'(x)^2 + g'(x)^2 = \begin{cases} 4x^2 + 1 & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

(c) Is there a differentiable function h such that $h'(x) = f'(x)^2 + g'(x)^2$ for all $x \in \mathbb{R}$?

Problem 5 (10 + 5 = 15 points)

(a) Let $x > 4$. Show that there exists $c \in (4, x)$ such that

$$\sqrt{x} = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{16c^{5/2}}(x-4)^3.$$

(b) Let $r = 2 + \frac{1}{4} - \frac{1}{64} = \frac{143}{64}$. Show that $0 < \sqrt{5} - r < \frac{1}{512}$.

Problem 6 (5 + 10 = 15 points)

(a) Assume that $f : \mathbb{R} \to \mathbb{R}$ is differentiable. Use the Mean Value Theorem to prove that if $f'(x) = 0$ for all $x \in \mathbb{R}$, then f is constant.

(b) Define the function $f : \mathbb{R} \to \mathbb{R}$ as

$$f(x) = \int_x^{x+2\pi} \ln(2 + \cos(t))\, dt.$$

Prove that f is constant.

End of test ("version 1", 90 points)

Version 2 (for even student numbers)

Problem 1 (5 + 5 + 5 = 15 points)

Given an example of each of the following, or argue that such a request is impossible:

(a) A nonempty set with only finitely many elements that does *not* contain its infimum.

(b) A nonempty bounded subset of $\mathbb{Q}$ that contains its supremum but not its infimum.

(c) A nonempty set S with the property $\inf S \geq \sup S$.

Problem 2 (6 + 6 + 3 = 15 points)

Assume that $-1 < r < 1$ and $c \in \mathbb{R}$. Consider the sequence (x_n) defined inductively as follows:

$$x_{n+1} = rx_n + c, \quad n \in \mathbb{N},$$

where $x_1 \in \mathbb{R}$ is arbitrary.

(a) Prove by induction that $x_{n+1} = r^n x_1 + \sum_{k=0}^{n-1} cr^k$ for all $n \in \mathbb{N}$.

(b) Prove that the sequence (x_n) converges.

(c) Compute $\lim x_n$ when $r = 1/2$ and $c = 5$.

Problem 3 (15 points)

Let $A \subseteq \mathbb{R}$ be a nonempty set. Prove that the following set is compact:

$$K = \bigcap_{a \in A} [a-1, a+1].$$

Problem 4 (5 + 5 + 5 = 15 points)

Consider the following functions:

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases} \quad \text{and} \quad g(x) = \begin{cases} x^2 \cos(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

(a) Prove that f and g are differentiable at $x = 0$ and that $f'(0) = g'(0) = 0$.
Hint: only give the proof for f, since the proof for g is similar.

(b) Show that

$$f'(x)^2 + g'(x)^2 = \begin{cases} 4x^2 + 1 & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

(c) Is there a differentiable function h such that $h'(x) = f'(x)^2 + g'(x)^2$ for all $x \in \mathbb{R}$?

Problem 5 (10 + 5 = 15 points)

(a) Let $x > 4$. Show that there exists $c \in (4, x)$ such that

$$\sqrt{x} = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{16c^{5/2}}(x-4)^3.$$

(b) Let $r = 2 + \frac{1}{2} - \frac{1}{16} = \frac{39}{16}$. Show that $0 < \sqrt{6} - r < \frac{1}{64}$.

Problem 6 (5 + 10 = 15 points)

(a) Assume that $f : \mathbb{R} \to \mathbb{R}$ is differentiable. Use the Mean Value Theorem to prove that if $f'(x) = 0$ for all $x \in \mathbb{R}$, then f is constant.

(b) Define the function $f : \mathbb{R} \to \mathbb{R}$ as

$$f(x) = \int_x^{x+2\pi} e^{\sin(t)}\, dt.$$

Prove that f is constant.

End of test ("version 2", 90 points)

Solution of Problem 1 (5 + 5 + 5 = 15 points)

(a) *Version 1.* This request is impossible. If a set is finite, then the supremum equals the largest value of all elements in the set and hence is contained in the set.
(5 points)

Version 2. This request is impossible. If a set is finite, then the infimum equals the smallest value of all elements in the set and hence is contained in the set.
(5 points)

(b) *Version 1.* An example of such a set is given by

$$S = \{x \in \mathbb{Q} : 0 \leq x \leq \sqrt{2}\}.$$

Clearly, $\inf S = 0$ is contained in S. However, $\sup S = \sqrt{2}$ is not contained in S.
(5 points)

Version 2. An example of such a set is given by

$$S = \{x \in \mathbb{Q} : \sqrt{2} \leq x \leq 2\}.$$

Clearly, $\sup S = 2$ is contained in S. However, $\inf S = \sqrt{2}$ is not contained in S.
(5 points)

(c) An example of such a set is $S = \{0\}$. Then $\inf S = 0$ and $\sup S = 0$, and clearly $\inf S \geq \sup S$.
(5 points)

In fact, the only possible example is a set which contains precisely one element.

Solution of Problem 2 (6 + 6 + 3 = 15 points)

(a) For $n = 1$ we have that

$$r^n x_1 + \sum_{k=0}^{n-1} cr^k = r^1 x_1 + \sum_{k=0}^{0} cr^k = rx_1 + c = x_2,$$

where the last equality sign follows from the definition of the sequence (x_n). Therefore, the formula holds for $n = 1$.

(1 point)

Assume that the formula holds for *some* $n \in \mathbb{N}$. Then we have that

$$\begin{aligned} x_{n+2} &= rx_{n+1} + c \\ &= r\left(r^n x_1 + \sum_{k=0}^{n-1} cr^k\right) + c \\ &= r^{n+1} x_1 + \sum_{k=0}^{n-1} cr^{k+1} + c \\ &= r^{n+1} x_1 + \sum_{k=1}^{n} cr^k + c = r^{n+1} x_1 + \sum_{k=0}^{n} cr^k, \end{aligned}$$

which shows that the formula also holds for $n + 1$. By induction, the formula holds for all $n \in \mathbb{N}$.

(5 points)

(b) Note that

$$s_n = \sum_{k=0}^{n-1} cr^k$$

is the partial sum of a convergent geometric series. Therefore, (s_n) is convergent.

(2 points)

We have that

$$x_{n+1} = r^n x_1 + s_n$$

and $\lim r^n = 0$ since $-1 < r < 1$.

(2 points)

Since (x_{n+1}) converges if and only if (x_n) converges, it follows from the Algebraic Limit Theorem that (x_n) converges.

(2 points)

(c) From part (a) and the theory of geometric series obtain that

$$\lim x_n = \lim s_n = \sum_{k=0}^{\infty} cr^k = \frac{c}{1-r}.$$

(2 points)

This gives

$$\lim x_n = \begin{cases} \dfrac{4}{1+1/3} = 3 & \text{for version 1,} \\ \dfrac{5}{1-1/2} = 10 & \text{for version 2.} \end{cases}$$

(1 point)

Solution of Problem 3 (15 points)

Note that K could be empty. For example, this happens when $A = \{0, 3\}$ in which case

$$K = [-1, 1] \cap [2, 4] = \varnothing.$$

But the empty set is compact.
(3 points)

Note that K is closed since it is an intersection of closed sets.
(4 points)

The set K is also bounded. Indeed, since A is nonempty there exists at least one element $a_0 \in A$. Then we have

$$K \subset [a_0 - 1, a_0 + 1],$$

which implies that K is bounded.
(4 points)

Since K is closed and bounded it follows that K is compact.
(4 points)

Solution of Problem 4 (5 + 5 + 5 = 15 points)

(a) If $x \neq 0$, then we have

$$\left|\frac{f(x)-f(0)}{x-0} - 0\right| = |x\sin(1/x)| = |x|\,|\sin(1/x)| \leq |x| = |x-0|.$$

(2 points)

Let $\epsilon > 0$ be arbitrary and take $\delta = \epsilon$. If $0 < |x-0| < \delta$, then

$$\left|\frac{f(x)-f(0)}{x-0} - 0\right| < \epsilon.$$

This implies that f is differentiable at $x = 0$ and $f'(0) = 0$.
(3 points)

(b) From part (a) it follows that $h'(0) = f'(0)^2 + g'(0)^2 = 0$. For $x \neq 0$ we can differentiate f and g using the rules from calculus:

$$\begin{aligned} f'(x) &= 2x\sin(1/x) - \cos(1/x), \\ g'(x) &= 2x\cos(1/x) + \sin(1/x). \end{aligned}$$

(3 points)

Taking squares gives

$$\begin{aligned} f'(x)^2 &= 4x^2\sin^2(1/x) - 4x\sin(1/x)\cos(1/x) + \cos^2(1/x), \\ g'(x)^2 &= 4x^2\cos^2(1/x) + 4x\cos(1/x)\sin(1/x) + \sin^2(1/x), \end{aligned}$$

so that

$$f'(x)^2 + g'(x)^2 = 4x^2 + 1.$$

(2 points)

(c) Assume there exists a function h such that $h'(x) = f'(x)^2 + g'(x)^2$. Note that $h'(0) = 0$ and $h'(1) = 5$. By Darboux's Theorem it follows that for any $\alpha \in (0,5)$ there exists $c \in (0,1)$ such that $h'(c) = \alpha$. In particular, this must be the case for $\alpha = 1/2$. However, $f'(x)^2 + g'(x)^2 \geq 1$ for all $x \neq 0$. We conclude that such a function h does not exist.
(5 points)

Solution of Problem 5 (10 + 5 = 15 points)

(a) For $f(x) = \sqrt{x} = x^{1/2}$ we have

$$f'(x) = \tfrac{1}{2}x^{-1/2}, \quad f''(x) = -\tfrac{1}{4}x^{-3/2}, \quad f'''(x) = \tfrac{3}{8}x^{-5/2}.$$

(4 points)

Evaluating at $x = 4$ gives

$$f(4) = 2, \quad f'(4) = \tfrac{1}{4}, \quad f''(4) = -\tfrac{1}{32}.$$

(4 points)

If $x > 4$, then the Lagrange Remainder Theorem with $n = 2$ and $a = 4$ gives the existence of a point $c \in (4, x)$ such that

$$\begin{aligned}\sqrt{x} &= f(4) + f'(4)(x-4) + \tfrac{1}{2}f''(4)(x-4)^2 + \tfrac{1}{6}f'''(c)(x-4)^3 \\ &= 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{16c^{5/2}}(x-4)^3.\end{aligned}$$

(2 points)

(b) *Version 1.* Taking $x = 5$ in part (a) gives the existence of a point $c \in (4, 5)$ such that

$$\sqrt{5} - \left(2 + \frac{1}{4} - \frac{1}{64}\right) = \frac{1}{16c^{5/2}}.$$

The right hand side is obviously positive, which shows that $0 < \sqrt{5} - r$.
(2 points)

Since $c \in (4, 5)$ we have that

$$\frac{1}{16c^{5/2}} < \frac{1}{16 \cdot 4^{5/2}} = \frac{1}{16 \cdot 32} = \frac{1}{512},$$

which shows that $\sqrt{5} - r < \frac{1}{512}$.
(3 points)

Version 2. Taking $x = 6$ in part (a) gives the existence of a point $c \in (4, 6)$ such that

$$\sqrt{6} - \left(2 + \frac{1}{2} - \frac{1}{16}\right) = \frac{1}{2c^{5/2}}.$$

The right hand side is obviously positive, which shows that $0 < \sqrt{6} - r$.
(2 points)

Since $c \in (4, 6)$ we have that

$$\frac{1}{2c^{5/2}} < \frac{1}{2 \cdot 4^{5/2}} = \frac{1}{2 \cdot 32} = \frac{1}{64},$$

which shows that $\sqrt{6} - r < \frac{1}{64}$.
(3 points)

Solution of Problem 6 (5 + 10 = 15 points)

(a) The Mean Value Theorem implies that for every $x \neq 0$ there exists a point c between 0 and x such that

$$\frac{f(x) - f(0)}{x - 0} = f'(c).$$

(2 points)

Since the derivative of f is assumed to be zero everywhere, it follows that

$$f(x) - f(0) = 0$$

for all $x \neq 0$. This implies that $f(x) = f(0)$ for all $x \in [0, 1]$, which means that f is constant.
(3 points)

(b) We can rewrite the function F as follows:

$$f(x) = \begin{cases} \displaystyle\int_0^{x+2\pi} \ln(2 + \cos(t))\,dt - \int_0^{x} \ln(2 + \cos(t))\,dt & \text{for version 1,} \\ \displaystyle\int_0^{x+2\pi} e^{\sin(t)}\,dt - \int_0^{x} e^{\sin(t)}\,dt & \text{for version 2.} \end{cases}$$

(4 points)

Since the integrand is a continuous function, we are allowed to use the second part of the Fundamental Theorem of Calculus. This gives

$$f'(x) = \begin{cases} \ln(2 + \cos(x + 2\pi)) - \ln(2 + \cos(x)) & \text{for version 1,} \\ e^{\sin(x+2\pi)} - e^{\sin(x)} & \text{for version 2.} \end{cases}$$

(2 points)

Since the cosine and sine functions are 2π-periodic, it follows that $f'(x) = 0$ for all $x \in \mathbb{R}$.
(2 points)

Since the derivative of f is zero, it follows that f itself is constant by part (a).
(2 points)